Example Using the limit of slopes, find the eqn of the tangent line to
$$y = x - 2 - \cos(2x) \text{ at } x = \frac{\pi}{6}.$$

Solution

- Find deriv at $x = \frac{\pi}{6}$ ← let $m =$
- $y - y_0 = m(x - x_0)$ gives eqn of line.

$$f(x) = x - 2 - \cos(2x)$$
$$a = \frac{\pi}{6}$$

$$y_0 = f(a) = \frac{\pi}{6} - 2 - \frac{1}{2} = \frac{\pi}{6} - \frac{5}{2}$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\left(\frac{\pi}{6} + h\right) - 2 - \cos\left(2\left(\frac{\pi}{6} + h\right)\right)\right) - \left(\frac{\pi}{6} - 2 - \cos\left(2 \cdot \frac{\pi}{6}\right)\right)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{\frac{\pi}{6}} + h \cancel{- 2} - \cos\left(\frac{\pi}{3} + 2h\right) - \cancel{\frac{\pi}{6}} \cancel{+ 2} + \overbrace{\cos\left(\frac{\pi}{3}\right)}^{1/2}}{h}$$

$$= \lim_{h \to 0} \frac{h - \cos\left(\frac{\pi}{3} + 2h\right) + \frac{1}{2}}{h}$$

$$\cos(A+B) = \cos(A)\cos(B) - \sin A \sin B$$

$$= \lim_{h \to 0} \frac{h - \left[\overset{1/2}{\cos\left(\frac{\pi}{3}\right)} \cos(2h) - \overset{\sqrt{3}/2}{\sin\left(\frac{\pi}{3}\right)} \sin(2h)\right] + \frac{1}{2}}{h}$$

$$= \lim_{h \to 0} \frac{h - \left[\frac{1}{2}\cos(2h) - \frac{\sqrt{3}}{2}\sin(2h)\right] + \frac{1}{2}}{h}$$

$$= \lim_{h \to 0} \frac{h - \frac{1}{2}\cos(2h) + \frac{\sqrt{3}}{2}\sin(2h) + \frac{1}{2}}{h}$$

$$= \lim_{h \to 0} \underbrace{\frac{h}{h}}_{\to 1} + \frac{\sqrt{3}}{2} \underbrace{\frac{\sin(2h)}{2h}}_{\to 1} \cdot \frac{2}{1} + \frac{\overset{\frac{1}{2}(1-\cos(2h))}{\frac{1}{2} - \frac{1}{2}\cos(2h)}}{h}$$

$$= \lim_{h \to 0} 1 + \sqrt{3} + \frac{\frac{1}{2}\left(1 - \cos(2h)\right)}{h}$$

$$\begin{aligned} \cos(2h) &= \cos^2(h) - \sin^2(h) \\ &= 1 - 2\sin^2(h) \\ &= 2\cos^2(h) - 1 \end{aligned}$$

$$= \lim_{h \to 0} 1 + \sqrt{3} + \frac{\frac{1}{2}\left(1 - \left(1 - 2\sin^2(h)\right)\right)}{\underset{\cancel{1} - \cancel{1} + 2\sin^2(h)}{h}}$$

$$= 1 + \sqrt{3} + \lim_{h \to 0} \cancel{\tfrac{1}{2}} \frac{\cancel{2}\sin^2(h)}{h}$$

$$= 1 + \sqrt{3} + \lim_{h \to 0} \underbrace{\frac{\sin(h)}{h}}_{\to 1} \cdot \underbrace{\sin(h)}_{\to 0}$$

$$= \boxed{1 + \sqrt{3}} = m$$

line equation

$$y - y_0 = m(x - x_0)$$

$$y - \left(\frac{\pi}{6} - \frac{5}{2}\right) = (1+\sqrt{3})\left(x - \frac{\pi}{6}\right)$$

$$y = (1+\sqrt{3})x - \frac{\pi}{6}(1+\sqrt{3}) + \frac{\pi}{6} + \frac{5}{2}$$

$$\boxed{y = (1+\sqrt{3})x - \frac{\pi\sqrt{3}}{6} + \frac{5}{2}}$$

<u>Example</u>: Find the formula for the derivative of $\sin(x)$ at any point x.

<u>Solution</u>: $f(x) = \sin(x)$

$$f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \underbrace{\frac{\sin(h)\cos(x)}{h}}_{1} + \frac{\sin(x)\cos(h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} 1 \cdot \cos(x) + \frac{\sin(x)(\cos(h) - 1)}{h}$$

$$= \lim_{h\to 0} \cos(x) + \frac{\sin(x)(\cos(h)-1)(\cos(h)+1)}{h(\cos(h)+1)}$$

$$= \lim_{h\to 0} \cos(x) + \frac{\sin(x)(\cos^2(h) - 1)}{h(\cos(h)+1)}$$

$\sin^2 + \cos^2 = 1$

$\cos^2 - 1 = -\sin^2$

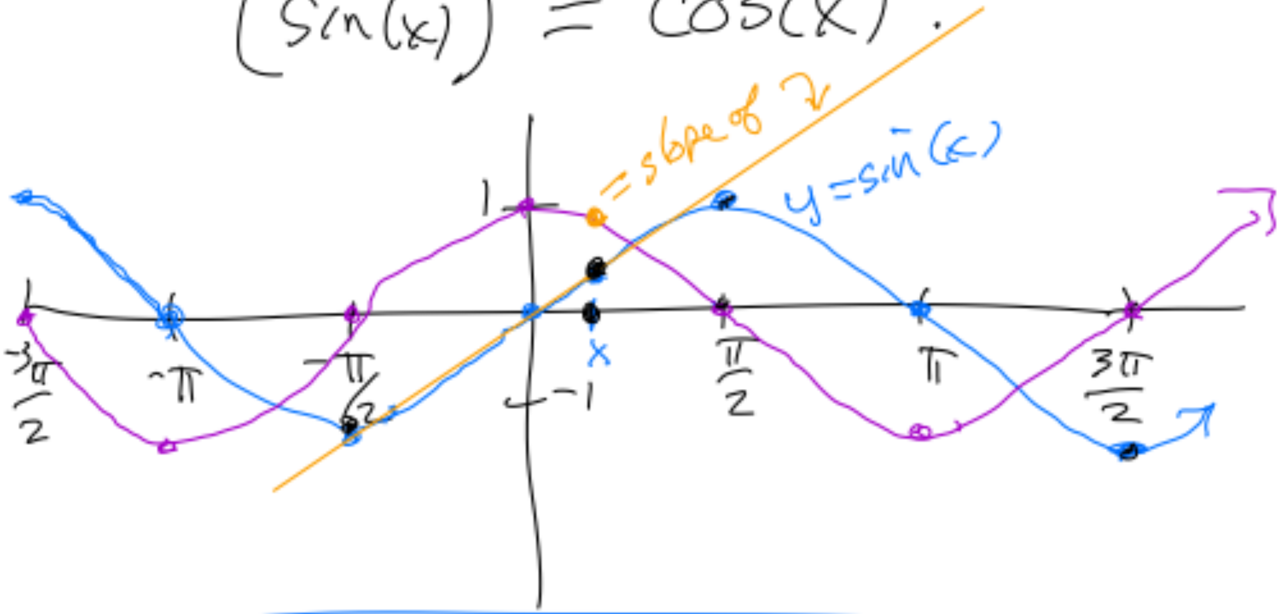
$$= \lim_{h\to 0} \cos(x) + \frac{\sin(x)}{h}\left(\frac{-\sin^2(h)}{\cos(h)+1}\right)$$

$$= \lim_{h\to 0} \cos(x) + -\sin(x) \cdot \underbrace{\frac{\sin(h)}{h}}_{1} \cdot \frac{\overbrace{\sin(h)}^{0}}{\underbrace{\cos(h)+1}_{2}}$$

$$= \boxed{\cos(x)}$$

Summary.

$$\left(\sin(x)\right)' = \cos(x).$$



at each point $(x, \sin(x))$ of the curve $y = \sin(x)$, the slope of the tangent line is $\cos(x)$.

Example Calculate the slope of the tangent line to $y = \sqrt{4x+1}$ at $x = 0$ (using a limit of slopes)

Solution: $f(x) = \sqrt{4x+1}$

$$f'(0) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{4h+1} - \sqrt{1}}{h}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{4h+1} - 1\right)}{h} \frac{\left(\sqrt{4h+1} + 1\right)}{\left(\sqrt{4h+1} + 1\right)}$$

$$= \lim_{h \to 0} \frac{(4h+1) - 1}{h\left(\sqrt{4h+1} + 1\right)}$$

$$= \lim_{h \to 0} \frac{4\cancel{h} \cancel{+1} \cancel{-1}}{\cancel{h}\underbrace{\left(\sqrt{4h+1} + 1\right)}_{2}} = \frac{4}{2} = \boxed{2}$$

Thus, the slope of the tangent line to $y = \sqrt{4x+1}$ at $x = 0$ is 2.

Let's do a quick check.

$f(x) = \sqrt{4h+1}$ close to $x=0$

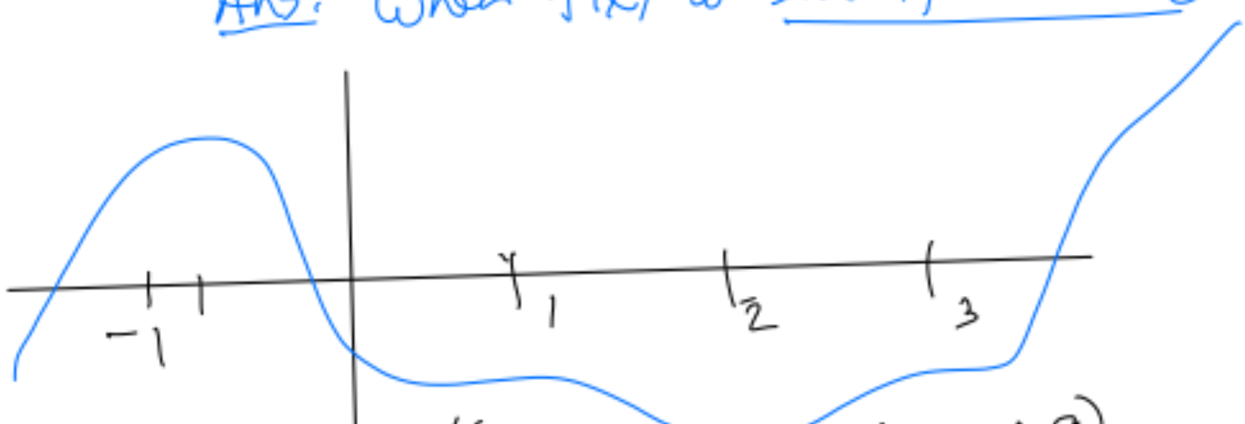
x	$f(x)$
0	1
0.001	$\sqrt{1.004} = 1.0019$

$$\text{slope} \approx \frac{\Delta y}{\Delta x} = \frac{1.0019 - 1}{0.001 - 0}$$

$$= \frac{.0019}{.001} = 1.9 \leftarrow \text{close to } 2.$$

If $y = f(x)$ is a function, when is $f'(x) > 0$?

<u>Ans:</u> when $f(x)$ is <u>strictly increasing</u>



$f'(x) > 0$ when $x \in (-2, -0.8)$,

also when $x > 3.2$.